PROJECTED WRITTEN NOTES FROM THE M408D LECTURE
ON THURSDAY, APRIL 18, 2024, ON SECTION 15.3 -
THE DOUBLE INTEGRAL IN POLAR COORDINATES

CLASS #26

The Double Integral over Regions in the Plane

Properties of the Double Integral

① $\iint_D r\cdot f(x,y)\,dA = r\iint_D f(x,y)\,dA$

$$\iint_D [f(x,y) \pm g(x,y)]\,dA = \iint_D f(x,y)\,dA \pm \iint_D g(x,y)\,dA$$

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② If $D =$ 

, then $\iint_D f(x,y)\,dA$

$$= \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

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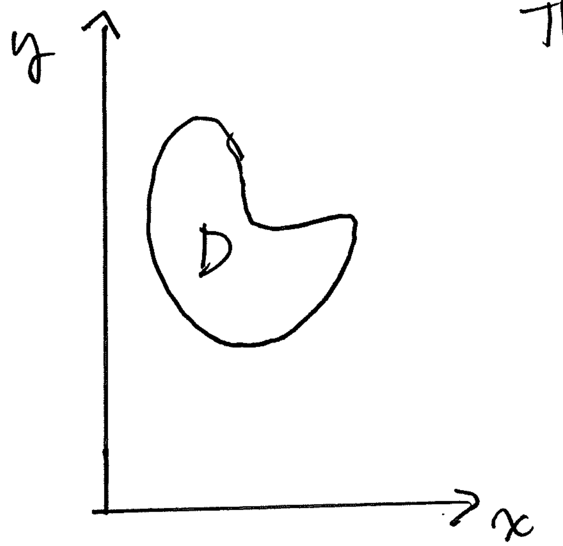
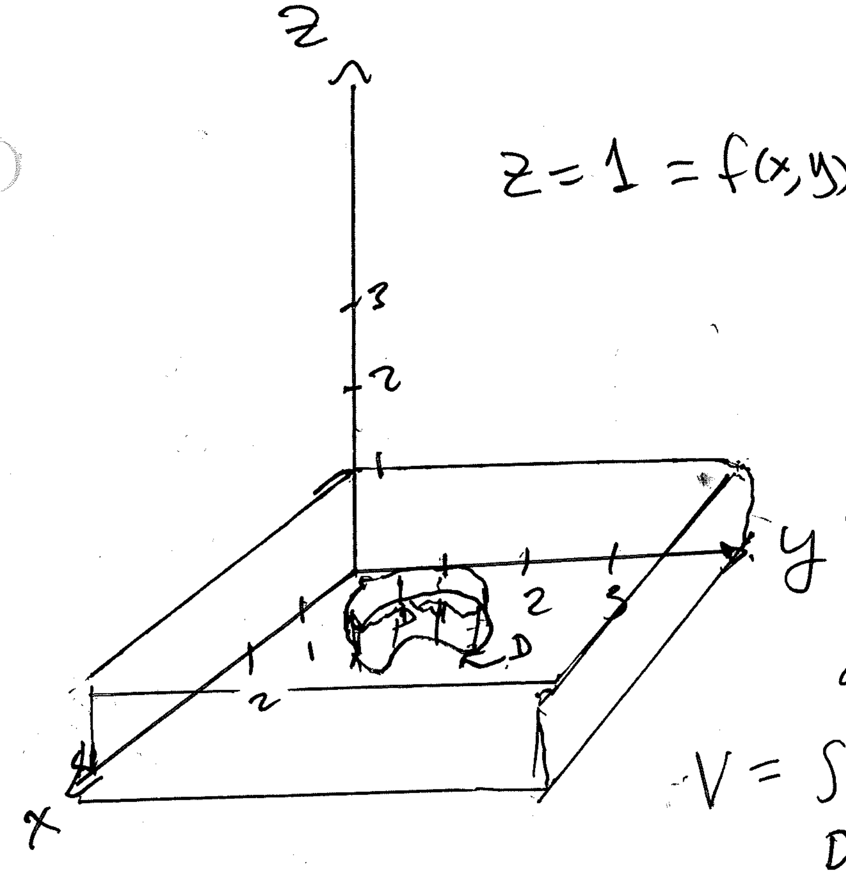
③



The Area A of Region D is $A = \iint_D 1\,dA$ square units

$z = f(x,y) = 1 + 0x + 0y$



$z = 1 = f(x,y)$

The Volume V of the Solid Region under the plane $z = 1$ and above Region D is

$V = \iint_D 1\,dA = (\text{Area of } D) \times 1$ cubic units

So, The Area of D $= \iint_D 1\,dA$ square units

Properties of Double Integrals

We assume that all of the following integrals exist. For rectangular regions D the first three properties can be proved in the same manner as in Section 5.2. And then for general regions the properties follow from Definition 2.

$$\boxed{5} \qquad \iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

$$\boxed{6} \qquad \iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

$$\boxed{7} \qquad \iint_D f(x, y)\, dA \geqslant \iint_D g(x, y)\, dA$$

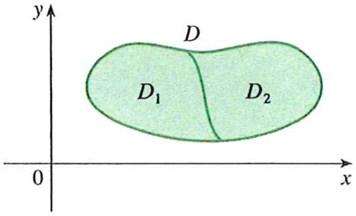


FIGURE 17

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The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ (Property 5 in Section 5.2).

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

$$\boxed{8} \qquad \boxed{\iint_D f(x, y)\, dA = \iint_{D_1} f(x, y)\, dA + \iint_{D_2} f(x, y)\, dA}$$

Property 8 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 67 and 68.)

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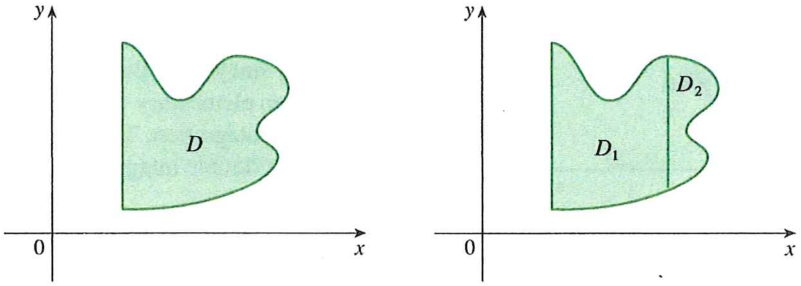

FIGURE 18

(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

$$\boxed{9} \qquad \boxed{\iint_D 1\, dA = A(D)}$$

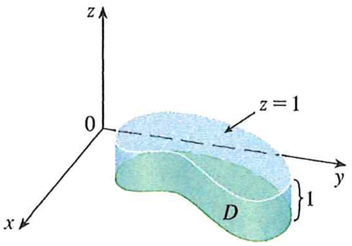

FIGURE 19
Cylinder with base D and height 1

Figure 19 illustrates why Equation 9 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 6, 7, and 9 to prove the following property. (See Exercise 73.)

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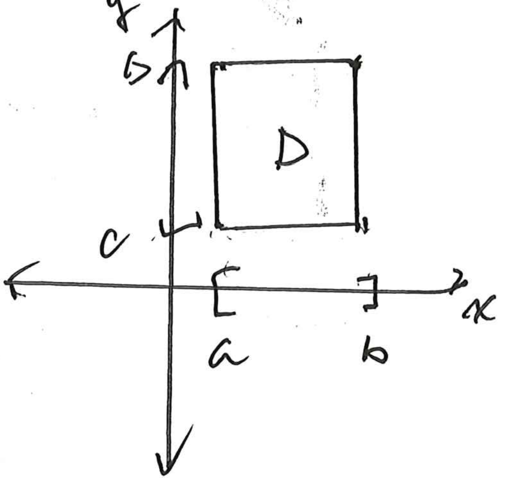
The Double Integral in Polar Coordinates

A Cartesian (x,y) Rectangle D is

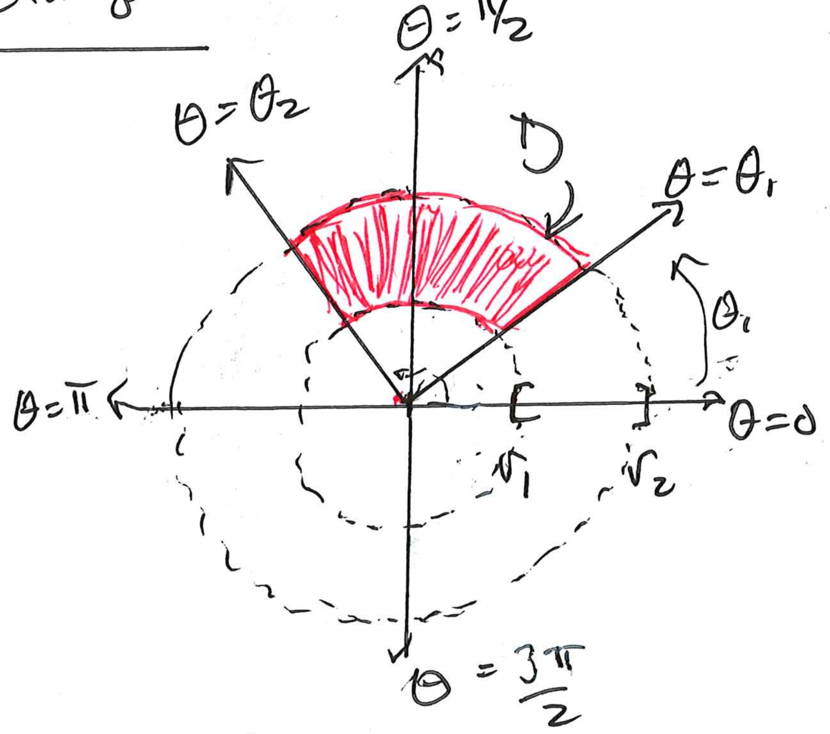


$D: [a,b] \times [c,d]$

D = all $(x,y)_R$ with

$a \le x \le b$ and $c \le y \le d$

A Polar (r,θ) Rectangle D



$D: [r_1, r_2] \times [\theta_1, \theta_2]$

$$D = \left\{ \text{all } (r,\theta)_P \;\middle|\; \begin{array}{c} \theta_1 \le \theta \le \theta_2 \\ \text{and} \\ r_1 \le r \le r_2 \end{array} \right\}$$

FIND $\iint_D f(x,y)\,dA$ using Polar Coordinates when D is a polar Rectangle..

The Situation: We are given $z = f(x,y)$ ← Rectangular Coord's

but $D = [a,b] \times [c,d]$ in Polar Coord's.

r's θ's

Recall, when $(x,y)_R$ and $(r,\theta)_P$ describe the same point:

$x = r\cos\theta$ $y = r\sin\theta$

$x^2+y^2 = r^2$ $y/x = \tan\theta$

So, $f(x,y) = f(r\cos\theta, r\sin\theta)$ ← Use this in the Integrand

we must also include a special r factor in the integrand.

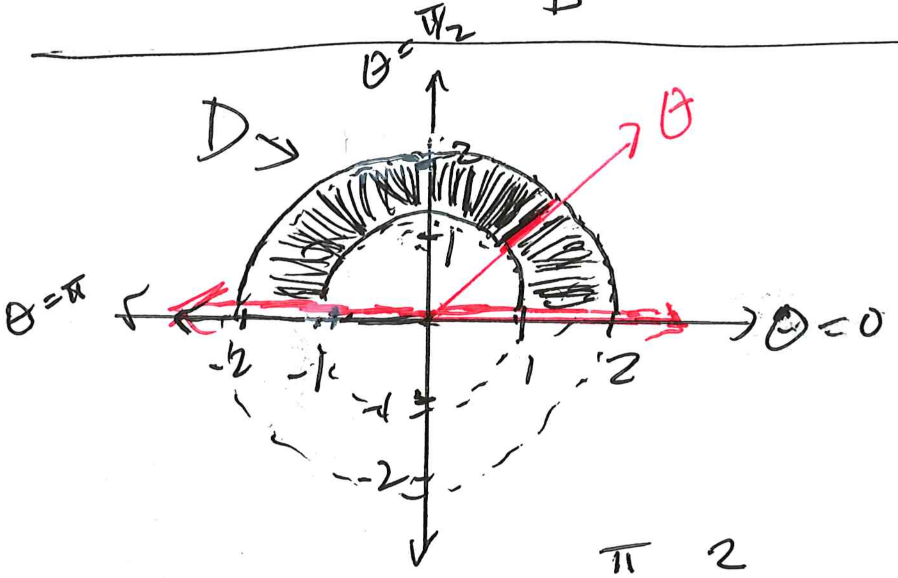
The formula for Integrating in Polar Coord's over a Polar RECTANGLE $D = [a,b] \times [\alpha,\beta]$ is

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) \cdot \underset{\nwarrow \text{The special factor } r.}{r} \cdot dr\,d\theta$$

With Polar Coord's, always write the iterated integral using "$dr\,d\theta$".

Ex. Consider the Polar Rectangle D where D: all $(r,\theta)_p$ with $0 \le \theta \le \pi$ and $1 \le r \le 2$.

<u>TASK:</u> Find $\iint_D (3x+4y^2)\,dA$.



$$\iint_D (3x+4y^2)\,dA = \int_0^\pi \int_1^2 \left(3r\cos\theta + 4(r\sin\theta)^2\right) \overset{\substack{\text{Special}\\ \text{FACTOR}\\ \downarrow}}{r}\,dr\,d\theta$$

$$= \int_0^\pi \int_1^2 \left(3r\cos\theta + 4r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \int_0^\pi \int_1^2 \left(3r^2\cos\theta + 4r^3\sin^2\theta\right) dr\,d\theta$$

$$= \int_0^\pi \left(\left(r^3\cos\theta + r^4\sin^2\theta\right)\Big|_{r=1}^{r=2} \right) d\theta$$

$$= \int_0^{\pi} \Big((8\cos\theta + 16\sin^2\theta) - (\cos\theta + \sin^2\theta) \Big)\, d\theta$$

$$= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta \quad \longleftarrow \left(\text{use } \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \right)$$

$$= \int_0^{\pi} \left(7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos 2\theta \right) d\theta$$

$$= \left(7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \right) \Big|_0^{\pi}$$

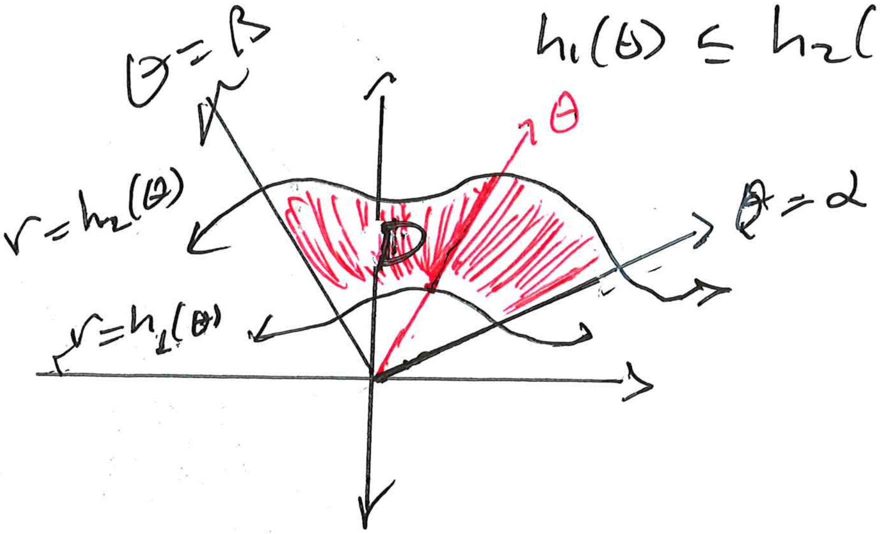
$$= \left(0 + \frac{15}{2}\pi - 0 \right) - (0)$$

$$= \frac{15\pi}{2}$$

$$\iint_D (3x + 4y^2)\, dA = \frac{15\pi}{2}.$$

A Type II Polar Region

New Situation: We are given an interval $[\alpha, \beta]$ and two functions of θ, $r = h_1(\theta)$ and $r = h_2(\theta)$ such that $\alpha < \beta$ and $h_1(\theta) \leq h_2(\theta)$ for all $\alpha \leq \theta \leq \beta$.

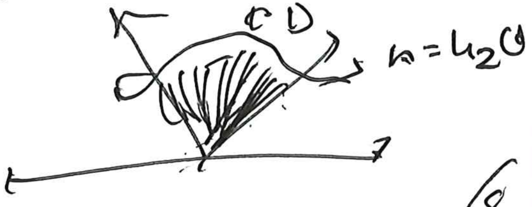


D: all $(r,\theta)_p$ such that $\alpha \leq \theta \leq \beta$ and $h_1(\theta) \leq r \leq h_2(\theta)$.

Given $z = f(x,y)$,

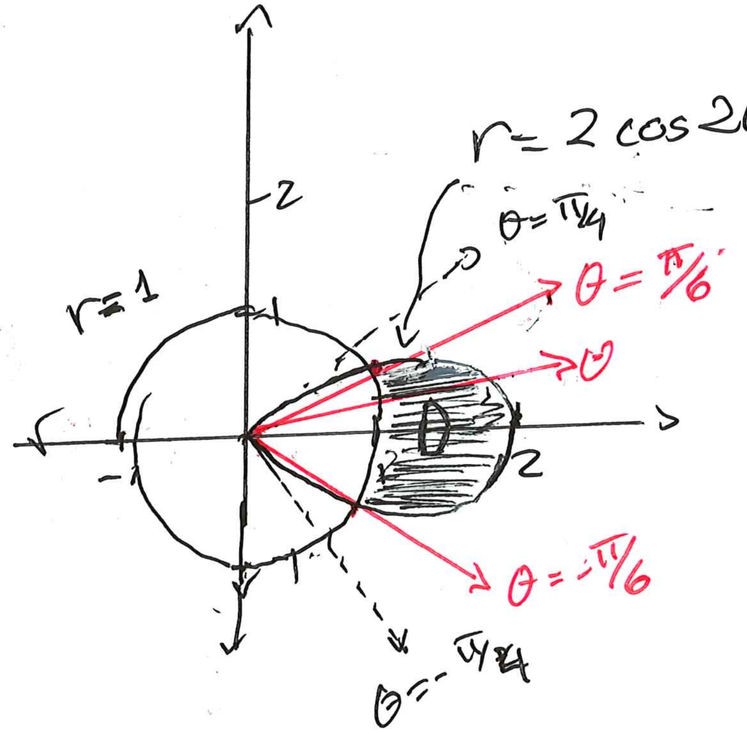
$$\iint_D f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\cdot r\cdot dr\,d\theta$$

NOTE: $r = h_1(\theta)$ might be $r = 0$, which means the Region D extends all the way Down to the Pole $(0,0)$.



A PROBLEM Evaluating a Double Integral over a Type II Polar Region.

Problem: D is the region inside the Petal $r = 2\cos 2\theta$, $-\pi/4 \le \theta \le \pi/2$, and is OUTSIDE the circle $r = 1$.



TASK! Determine the Area of D by evaluating $\iint_D 1\,dA$

The Area of D $= \iint_D 1\,dA$

$$A = \int_{-\pi/6}^{\pi/6} \int_{1}^{2\cos 2\theta} 1 \cdot r\,dr\,d\theta = \ldots = \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) \text{sq units}$$

Problems : Change the Order of Integration

$$I = \int_0^2 \int_{x^2}^{4} f(x,y)\,dy\,dx$$

$\leftarrow y=4$

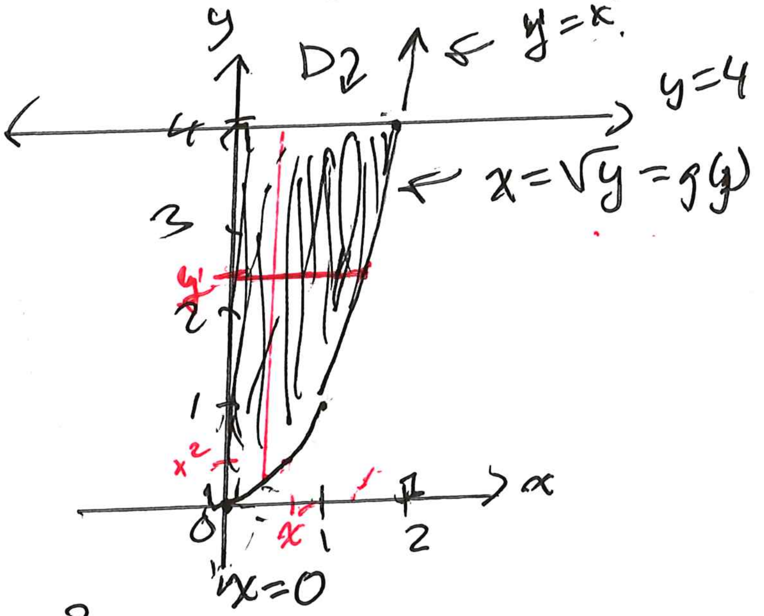
$\nwarrow y=x^2$

$\nwarrow$ Type I Region

A Type I Description of D:

D: $0 \le x \le 2$

and $x^2 \le y \le 4$



$y = x^2$, $\sqrt{y} = x$

$x \ge 0$ $\quad x \ge 0$

Type II Desc of D

D: $0 \le y \le 4$

$0 \le x \le \sqrt{y}$

$$I = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$

$\nwarrow$ Type II

$$\iint_D f(x,y)\,dA = I = \int_0^2 \int_{x^2}^{4} f(x,y)\,dy\,dx = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$$